

Does the generalized second law require entropy bounds for a charged system?

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(February 7, 2008)

Abstract

We calculate the net change in generalized entropy occurring when one carries out the gedanken experiment in which a box initially containing energy E , entropy S and charge Q is lowered adiabatically toward a Reissner-Nordström black hole and then dropped in. This is an extension of the work of Unruh-Wald to a charged system (the contents of the box possesses a charge Q). Their previous analysis showed that the effects of acceleration radiation prevent violation of the generalized second law of thermodynamics. In our more generic case, we show that the properties of the thermal atmosphere are equally important when charge is present. Indeed, we prove here that an equilibrium condition for the the thermal atmosphere and the physical properties

of ordinary matter are sufficient to enforce the generalized second law. Thus, no additional assumptions concerning entropy bounds on the contents of the box need to be made in this process. The relation between our work and the recent works of Bekenstein and Mayo [11], and Hod [12] (entropy bound for a charged system) are also discussed.

PACS number(s): 04.70.Dy

I. INTRODUCTION

One of the most remarkable developments in black hole physics is the relationship between the laws of black hole mechanics and thermodynamics. Classically, black holes obey the laws that are analogous to the ordinary laws of thermodynamics [1]. This correspondence becomes more than just an analogy when quantum effects are taken into account (Hawking's discovery of the thermal radiation emitted by a black hole [2]).

Furthermore, Bekenstein [3] has conjectured a generalized second law (GSL) of thermodynamics: The sum of the black hole entropy and the ordinary entropy of the matter outside the black hole never decreases. More precisely, the GSL states that the generalized entropy S_g defined by

$$S_g = S_{matter} + \frac{1}{4}A_{bh} \tag{1}$$

never decreases for any physical process (we use natural units such that $\hbar = G = c = k = 1$ throughout this paper), where S_{matter} is the entropy of ordinary matter outside the black hole and $A_{bh}/4$, one quarter of the surface area of the black hole, plays the role of the entropy of the black hole. It is important to check the validity of this conjecture because this would strongly support the idea that the ordinary laws of thermodynamics apply to a self-gravitating quantum system containing a black hole and that $A_{bh}/4$ truly represents the physical entropy of the black hole.

There currently exists no general proof of the GSL based on the known microscopic laws of physics, although there are some proofs that rely on the semiclassical approximation [4–6]. This is because the laws of quantum gravity are not well known. Thus, gedanken experiments to test the validity of the GSL are very important tools to bolster confidence in this conjecture.

Classically, It was already recognized that a promising possibility for achieving a violation of the GSL occurs when one slowly (adiabatically) lowers a box initially containing energy E and entropy S toward a black hole and then dropped in [3]. The energy delivered to the

black hole can be arbitrarily red-shifted by letting the dumping point approach the horizon. Near this limit, the black hole area increase is not large enough to compensate for the decrease of the matter's entropy. A resolution of this difficulty was proposed by Bekenstein, who conjectured that there exists a universal upper bound on the entropy S of matter with energy E which is placed in a box of size R [7]:

$$S \leq 2\pi ER. \quad (2)$$

The intuitive reason why such a bound could rescue the GSL is that it prevents one from lowering a box close enough to a black hole to violate the GSL.

However, Unruh and Wald [8,9] pointed out that Bekenstein failed to take into account certain quantum effects in his analysis. They noted that there is a quantum thermal atmosphere surrounding a black hole, which produces a buoyancy force on a box when one tries to lower the box slowly toward the black hole. As a result, one cannot lower the box down to the horizon (if one does not wish to inject energy by pushing it in) and the box will float at a finite distance from the horizon, which is determined by the condition that the energy contained in the box is exactly the same as the energy of the acceleration radiation displaced by the box. Since the total energy at infinity added to the black hole after the box has been dropped from the floating point is larger than the redshifted proper energy of the box, the box must be opened (this was extended to the “dropped” case, recently [10]) at the floating point in order to minimize the entropy increase of the black hole. Accordingly, they concluded that the GSL holds in this process provided only that unconstrained thermal matter maximizes entropy at fixed volume and energy:

$$S \leq Vs(e), \quad (3)$$

where $s(e)$ is the entropy density as a function of energy density e of unconstrained thermal matter. Thus, they concluded that no additional assumption on the quantum nature of the matter such as (2) is necessary to rescue the GSL.

Recently, Bekenstein and Mayo [11] and Hod [12] have derived an upper bound to the entropy of a charged system by considering the polarization of the black hole by a nearby

charge. They argued that the GSL could be saved only by assuming the existence of entropy bounds on confined systems of the type as stated above. In their derivation, they regard the system as a “point particle” and used the test particle approximation. That is, the system is assumed to follow the equation of motion of a charged particle on a black hole background and has a conserved energy (the “backreaction” effects are negligible). However, since the system does not descend slowly (adiabatically) to the black hole in this process, there must be backreaction effects: the system radiates gravitational and electromagnetic radiation (these process also carry entropy) and the generalized entropy should increase if all these effects are included. Further more, there is no justification for treating the system as a point particle: the thermodynamical properties in and outside the box are completely neglected, even though they play an important role in the validity of the GSL [8–10]. Thus, it is doubtful if this composite system can be considered to be thermal.

In order to avoid these difficulties, we carry out a gedanken experiment in which a (possibly “thick”) box initially containing energy E , entropy S and charge Q is lowered adiabatically toward a Reissner-Nordström black hole and then dropped in. This is an extension of the work of Unruh-Wald to a charged system (the contents of the box possess a charge Q). Their previous analysis showed that the effects of acceleration radiation (buoyancy force) prevent a violation of the GSL, as stated above. Here, in addition to adding charge to the box, we consider the more generic case in which the thermal atmosphere has a spherically distributed charge, too. In this case, we notice that, in addition to the Unruh-Wald entropy restriction, there is an equilibrium condition for the chemical potential of the thermal atmosphere. Indeed, we prove here that these two equilibrium conditions and the physical properties of ordinary matter are sufficient to enforce the generalized second law. Thus, no additional assumptions concerning entropy bounds on the contents of the box need to be made in this process.

In Sec.II, we derive the equations that hold for the thermal atmosphere around a black hole. In Sec.III, we show that the GSL holds in the aforementioned process. Sec.IV is devoted to a summary and discussion of our results and, in particular, comparison with

previous works [11,12].

II. THERMAL ATMOSPHERE AROUND A BLACK HOLE

We carry out a gedanken experiment with a Reissner-Nordström black hole of mass M and charge Q_{bh} , whose spacetime metric and electromagnetic vector potential are given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (4)$$

$$A_\mu dx^\mu = \Phi(r)dt \equiv -\frac{Q_{bh}}{r}dt, \quad (5)$$

where

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2}, \quad (6)$$

with $r_\pm \equiv M \pm \sqrt{M^2 - Q_{bh}^2}$. The event horizon is located at $r = r_+$ and has area $A = 4\pi r_+^2$.

The temperature of the black hole is defined by [2]

$$T_H = \frac{1}{2\pi}\kappa \equiv \frac{1}{4\pi}f'(r_+), \quad (7)$$

where f' denotes df/dr and $\kappa = (r_+ - r_-)/2r_+^2$ is the surface gravity of the black hole. Physically this represents the temperature of the black hole measured at infinity.

First, we give a definition of unconstrained thermal matter with charge. We define unconstrained thermal matter in a given region outside the black hole to be the state of matter that maximizes entropy at a fixed volume, energy and charge (electromagnetic potential given by (5)). Note that the properties of unconstrained thermal matter depend on location, i.e., the entropy density of unconstrained thermal matter, \tilde{s} , is a function of energy density, $\tilde{\rho}$, charge density, \tilde{q} , at the given point outside the black hole. We assume that the thermal atmosphere of a black hole is described by unconstrained thermal matter.

Then, the local temperature of a thermal atmosphere which is in equilibrium with the black hole is given by using the Tolman's law [13] as

$$\tilde{T} = T_H/\chi, \quad (8)$$

where $\chi = f^{1/2}$ is the redshift factor.

In addition to (8), the chemical potential of the thermal atmosphere $\tilde{\mu}_i$ must satisfy the following condition in order that it be in an equilibrium state [14]:

$$\tilde{\mu}_i \chi = \text{Constant for each } i, \quad (9)$$

where index i denotes particle species.

Following Unruh and Wald [8,9], we assume that the black hole has reached thermal equilibrium with the radiation, the whole system being enclosed in a large cavity. This is achieved by fixing the boundary condition at the boundary, i.e., by specifying the temperature and the electrostatic potential (these are determined by the Hawking radiation and the difference between the chemical potentials, respectively) at the boundary.

The first law for the thermal atmosphere is written as

$$d\tilde{\rho} = \tilde{T}d\tilde{s} + \tilde{q}d\phi + \sum_i \tilde{\mu}_i d\tilde{n}_i, \quad (10)$$

where $\phi = \Phi/\chi$. The integrated Gibbs-Duhem relation [15] for this system is as follows. (See appendix A for a derivation.)

$$\tilde{\rho} = \tilde{T}\tilde{s} - \tilde{P} + \sum_i \tilde{\mu}_i \tilde{n}_i. \quad (11)$$

Note that the quantity ϕ does not appear in this expression.

By using the above two equations, the following equation is derived from Eqs. (8) and (9).

$$d(\tilde{P}\chi) = -\tilde{\rho}d\chi - \tilde{q}\chi d\phi. \quad (12)$$

Eq. (12) states that pressure gradient is balanced by gravitational and electromagnetic forces.

III. VALIDITY OF THE GENERALIZED SECOND LAW

In this section, following [9,10,16], we compute the change in generalized entropy occurring when matter in a (possibly “large”) box is slowly lowered toward a black hole and then

dropped in. We consider a box of cross-sectional area A and height b , which contain energy density ρ , charge density q and total entropy S . As the box is lowered toward the black hole, the energy and charge density will depend both on the height l of the center of the box above the horizon, and the position within the box, y , as measured from the center.

We adopt the following notation for integrals [16]

$$\int f(y)dV \equiv A \int_{-b/2}^{b/2} f(y)dy. \quad (13)$$

The energy of the box as measured at infinity is

$$E_\infty(l) = \int \rho(l, y)\chi(l + y)dV, \quad (14)$$

whereas the gravitational and electromagnetic forces as measured at infinity are in the forms

$$w(l) = \int \rho(l, y)\frac{\partial\chi(l + y)}{\partial l}dV, \quad (15)$$

$$f_{em}(l) = \int q(l, y)\chi(l + y)\frac{\partial\phi(l + y)}{\partial l}dV. \quad (16)$$

These external forces do work on the gas in the box. We denote the work by $W_{ge}(l)$:

$$W_{ge}(l) = E_\infty(l) - E_i \quad (17)$$

$$= \int_\infty^l [w(l') + f_{em}(l')]dl', \quad (18)$$

where E_i is the initial energy of the box. This is equivalent to

$$dE_\infty = (w + f_{em})dl. \quad (19)$$

Meanwhile, the buoyancy force acting on the box, as measured at infinity, is equal to

$$f_b(l) = A \left[(\tilde{P}\chi)_{l-b/2} - (\tilde{P}\chi)_{l+b/2} \right], \quad (20)$$

where \tilde{P} is the radiation pressure of the thermal atmosphere. From Eq. (20), it is easy to show that the work done by the buoyancy force is given by

$$W_b(l) = \int_\infty^l f_b(l')dl' = \int \tilde{P}(l, y)\chi(l + y)dV. \quad (21)$$

Putting together Eqs. (17) and (21), the total work done on the box system is given by

$$W_{tot}(l) = W_{ge}(l) + W_b(l) \quad (22)$$

$$= \int [\rho(l, y) + \tilde{P}(l, y)] \chi(l + y) dV - E_i. \quad (23)$$

If the contents of the box are dropped into the black hole from position l_0 , the first law of black hole requires that the change ΔS_{bh} in black hole entropy should satisfy

$$\Delta S_{bh} = \frac{1}{T_H} (E_i + W_{tot}(l_0) - \Phi_{bh} Q) \quad (24)$$

$$= \frac{1}{T_H} \int [\rho(l_0, y) + \tilde{P}(l_0, y)] \chi(l_0 + y) dV - \frac{\Phi_{bh} Q}{T_H}, \quad (25)$$

where Q and $\Phi_{bh} = Q_{bh}/r_+$ are charge thrown into the system by the agent at infinity and electromagnetic potential of the black hole, respectively.

Hereafter, for simplicity, we consider 2-component system to be composed of a gas of particles with charge e and anti-particles with opposite charge $-e$. This assumption does not affect our result and it is easy to extend our argument to the $2n$ -component system¹, if we wish.

Therefore, by substituting Eqs. (8) and (11) into Eq. (25), we get the change in generalized entropy as

$$\Delta S_g = \Delta S_{bh} - S \quad (26)$$

$$= \frac{1}{T_H} \int [\rho(l_0, y) - \tilde{\rho}(l_0, y)] \chi(l_0 + y) dV + \tilde{S}(l_0) - S \\ + \frac{1}{T_H} \left\{ \sum_{i=+,-} \int \tilde{\mu}_i(l_0, y) \tilde{n}_i(l_0, y) \chi(l_0 + y) dV - \Phi_{bh} Q \right\}, \quad (27)$$

where $\tilde{S}(l_0) = \int \tilde{s}(l_0, y) dV$ and $S = \int s(l_0, y) dV$ are the entropy of the thermal atmosphere displaced by the box and the entropy of the matter in a box, respectively.

¹ If there is a particle with charge $e(> 0)$, there exist a corresponding anti-particle with opposite charge $-e$ in nature. So we consider an even number of particle species.

By using the equilibrium condition for the chemical potential of the thermal atmosphere (9), and noting that the chemical potential in the absence of the field can be neglected completely at the horizon because of its high (infinite) local temperature [18], we get

$$\begin{aligned}\tilde{\mu}_+ \tilde{n}_+ \chi + \tilde{\mu}_- \tilde{n}_- \chi &= \tilde{\mu}_+^h \chi_h \tilde{n}_+ + \tilde{\mu}_-^h \chi_h \tilde{n}_- \\ &= e \Phi_{bh} (\tilde{n}_+ - \tilde{n}_-),\end{aligned}\tag{28}$$

where the index h denotes the quantity evaluated at the horizon.

Thus, we can rewrite Eq.(27) further.

$$\begin{aligned}\Delta S_g &= \int \left\{ \tilde{s}(l_0) - \frac{1}{T_H} [\tilde{\rho}(l_0, y) \chi(l_0 + y) - \Phi_{bh} \tilde{q}(l_0, y)] \right\} dV \\ &\quad - \int \left\{ s(l_0) - \frac{1}{T_H} [\rho(l_0, y) \chi(l_0 + y) - \Phi_{bh} q(l_0, y)] \right\} dV,\end{aligned}\tag{29}$$

where $q = e(n_+ - n_-)$ and $\tilde{q} = e(\tilde{n}_+ - \tilde{n}_-)$ are the charge density of the matter in the box and that of the thermal atmosphere, respectively.

Since Eq. (19) can be rewritten as

$$\int \frac{\partial \rho(l, y)}{\partial l} \chi(l + y) dV - \int q(l, y) \chi(l + y) \frac{\partial \phi(l + y)}{\partial l} dV = 0,\tag{30}$$

it is easily shown by differentiating (27) that

$$\frac{\partial}{\partial l_0} \Delta S_{bh} = \frac{1}{T_H} \int \left\{ [\rho(l_0, y) - \tilde{\rho}(l_0, y)] \frac{\partial \chi(l_0 + y)}{\partial l_0} + [q(l_0, y) - \tilde{q}(l_0, y)] \chi(l_0 + y) \frac{\partial \phi(l_0 + y)}{\partial l_0} \right\} dV,\tag{31}$$

where we have used Eqs. (9) and (12), and the fact that total charge $Q = \int q dV$ of the particles in the box is conserved. Note that, since this process of lowering the box is adiabatic, no change in the entropy in the box can occur and thus ΔS_{bh} can be replaced by ΔS_g .

First, for simplicity, we consider the case in which the box is sufficiently “small” in the sense that the change in χ , $d\chi/dl$ and $d\Phi/dl$ across the box are small compared with their average values.

In this case, the floating point condition (31) and the total change in generalized entropy (29) reduce to

$$\rho(l_0) + q(l_0)\chi(l_0)\frac{\partial\phi(l_0)}{\partial\chi(l_0)} = \tilde{\rho}(l_0) + \tilde{q}(l_0)\chi(l_0)\frac{\partial\phi(l_0)}{\partial\chi(l_0)}, \quad (32)$$

and

$$\Delta S_g = \left\{ \tilde{S}(l_0) - \frac{1}{T_H} [\tilde{E}(l_0) - \tilde{Q}(l_0)\Phi_{bh}] \right\} - \left\{ S(l_0) - \frac{1}{T_H} [E(l_0) - Q(l_0)\Phi_{bh}] \right\}, \quad (33)$$

respectively. Here, we wrote $S = sV$, $E = \rho\chi V$, $Q = qV$ and the quantities with (\sim) refer to the thermal atmosphere.

Thus, our task is to seek the distribution function which maximizes the functional $S - (E - Q\Phi_{bh})/T_H$ under the constraint (32).

The state of matter is encoded in some density operator \hat{f} . By using it, we can express the energy density as $\rho = Tr(\hat{\rho}\hat{f})$ and charge density as $q = Tr(\hat{q}\hat{f})$, while entropy is defined by $sV = -Tr(\hat{f} \ln \hat{f})$. Then, Eq. (32) can be rewritten as

$$Tr[\hat{O}\hat{f}] = Tr[\hat{O}\tilde{f}], \quad (34)$$

where $\hat{O} \equiv \hat{\rho}\chi V + \hat{q}\chi^2 \frac{\partial\phi}{\partial\chi} V$, \hat{f} and \tilde{f} ($\propto \exp\{-(\tilde{H}_\infty - \tilde{Q}\Phi_{bh})/T_H\}$) denotes the density operator of the matter in a box and that of the thermal atmosphere which is in equilibrium with the black hole, respectively.

Considering that the variation of $S - (E - Q\Phi_{bh})/T_H$ under a small variation $\delta\hat{f}$ is given by

$$\begin{aligned} \delta[S - (E - Q\Phi_{bh})/T_H] &= -Tr[\delta\hat{f}(\ln \hat{f} + 1 + (\hat{\rho}\chi - \hat{q}\Phi_{bh})VT_H^{-1})] \\ &\equiv -Tr[\delta\hat{f}(\ln \hat{f} + 1 + (\hat{H}_\infty - \hat{Q}\Phi_{bh})T_H^{-1})], \end{aligned} \quad (35)$$

the functional $S - (E - Q\Phi_{bh})/T_H$ has an extremum under variations that preserve $Tr\hat{f} = 1$ and $Tr[\hat{O}\hat{f}] = Tr[\hat{O}\tilde{f}]$, where \hat{f} satisfies $(\ln \hat{f} + 1) + (\hat{H}_\infty - \hat{Q}\Phi_{bh})T_H^{-1} - \lambda_1 - \lambda_2\hat{O} = 0$. The quantities $\lambda_{1,2}$ are Lagrange multipliers for these constraints. Eliminating λ_1 by using $Tr\hat{f} = 1$, we get a unique solution

$$\hat{f} = \frac{1}{Z} \exp\left\{-\beta_H(\hat{H}_\infty - \hat{Q}\Phi_{bh}) + \lambda_2\hat{O}\right\}, \quad (36)$$

where $Z = \text{Tr}[\exp\{-\beta_H(\hat{H}_\infty - \hat{Q}\Phi_{bh}) + \lambda_2\hat{O}\}]$ and $\beta_H \equiv T_H^{-1}$. Then, by substituting Eq. (36) into Eq. (34), we get $\lambda_2 = 0$ and thus $\hat{f} = \tilde{f} = Z^{-1} \exp\{-\beta_H(\hat{H}_\infty - \hat{Q}\Phi_{bh})\}$.

Therefore, the maximum value of the functional $S - (E - Q\Phi_{bh})/T_H$ is realized for the thermal state with the canonical distribution $\hat{f} = Z^{-1} \exp\{-\frac{1}{T_H}(\hat{H}_\infty - \hat{Q}\Phi_{bh})\}$, which, in our case, corresponds to the thermal atmosphere of the black hole.

Hence, we have

$$\Delta S_g \geq \Delta S_g(l = l_0) \geq 0. \quad (37)$$

Thus, the GSL is satisfied in this process.

Next, we analyze the case of a “larger” box. The same procedure as for the “small” box can be applied in this case, too.

Hereafter, we adopt the following notation

$$\int a dV \equiv \text{Tr}[\hat{A}\hat{f}], \quad (38)$$

where a , \hat{A} and \hat{f} denote some observable, corresponding operator and density operator, respectively.

With this notation, the total change in generalized entropy (29) can be written in the form

$$\Delta S_g = U[\tilde{f}; \beta_H, \Phi_{bh}] - U[\hat{f}; \beta_H, \Phi_{bh}], \quad (39)$$

where U is a functional of a density matrix of the matter fields defined by

$$U[\hat{f}; \beta_H, \Phi_{bh}] \equiv -\text{Tr}[\hat{f} \ln \hat{f}] - \beta_H(\text{Tr}[\hat{H}_\infty \hat{f}] - \Phi_{bh} \text{Tr}[\hat{Q}\hat{f}]), \quad (40)$$

\hat{f} and \tilde{f} ($\propto \exp\{-\beta_H(\tilde{H}_\infty - \tilde{Q}\Phi_{bh})\}$) denote the density operators of the matter in the box and of the thermal atmosphere which is equilibrium with the black hole, respectively. In this expression, $\hat{H}_\infty \equiv \int \hat{\rho}\chi dV$ and $\hat{Q} \equiv \int \hat{q}dV$ are operators corresponding to energy (at infinity) and charge. Note that the functional U is essentially the negative of the free energy divided by the temperature.

Similarly, the floating point condition (31) can be reduced to

$$Tr[\hat{O}\tilde{f}] = Tr[\hat{O}\hat{f}] \quad (41)$$

$$\equiv \int (\rho \frac{\partial \chi}{\partial l} + q\chi \frac{\partial \phi}{\partial l}) dV. \quad (42)$$

These Eqs. (39) and (41) have just the same form as the Eqs. (33) and (34) for the “small” box. Therefore, by repeating the same procedure as in the “small” box’s case, we can show that violation of the GSL cannot be achieved in the case of a “large” box.

In obtaining these results, we have ignored any entropy emitted by the black hole. In fact, the entropy produced in spontaneous Unruh emission corresponding to the superradiant modes can be neglected by taking the black hole as a very massive one.

IV. SUMMARY AND DISCUSSION

We examined the gedanken experiment of lowering a box initially containing energy E , entropy S and charge Q toward a Reissner-Nordström black hole and then dropped in (an extension of the work of Unruh-Wald to the charged system). We have shown that the properties of the thermal atmosphere plays an important role in this case just as in Unruh-Wald’s case. Specifically, we used an assumption that unconstrained thermal matter maximizes entropy as a function of energy density and charge density, in addition to the Unruh-Wald buoyancy force. Note that an equilibrium condition for the chemical potential of the thermal atmosphere also plays an important role in this case. In deed, we proved here that these are sufficient for the enforcement of the GSL and no additional assumptions concerning entropy bounds on the contents of the box need to be made in this process.

Finally, we comment briefly on the relation between our work and the recent work of Bekenstein and Mayo [11], and Hod [12]. They have derived an upper bound to the entropy of a charged system by considering the polarization of the black hole by a nearby charge (gravitationally induced electrostatic self-force on a charged test particle [19]). They concluded that the GSL could be saved only by assuming the existence of entropy bounds on

a confined charged system. On the other hand, in our derivation, we have neglected the electrostatic self-force till now. If we want to include the electrostatic self-energy in our analysis, we have only to replace $\Phi \rightarrow \Phi \pm eM/2r^2$ in our analysis. The only effect is a change in Eq. (28), i.e.,

$$\tilde{\mu}_+ \tilde{n}_+ \chi + \tilde{\mu}_- \tilde{n}_- \chi = e\Phi_{bh}(\tilde{n}_+ - \tilde{n}_-) + \frac{e^2 M}{2r_+^2}(\tilde{n}_+ + \tilde{n}_-). \quad (43)$$

This correction gives a positive contribution to the net change in generalized entropy Eq. (29). Thus, in this gedanken experiment, the GSL would hold even if we include those self-interaction forces.

There are several advantages to our analysis compared with theirs. They regarded the system as a “point particle” (test particle approximation) and assumed that it follows the equation of motion of a charged particle on a black hole background and has a conserved energy (the “backreaction” effects are negligible). However, the system does not descend slowly (adiabatically) to the black hole in this process, the system would radiate gravitational and electromagnetic radiation (these process also carry entropy) and the generalized entropy should increase if all these effects are included. Of course, such an analysis including the backreaction effect would be too complicated to reach a definitive answer analytically. Compared this, since we very slowly (adiabatically) lowers the box toward the black hole (quasi-static process), these effects can be neglected. Furthermore, there is no justification for treating the system as a point particle: the thermodynamical properties in and outside the box is completely neglected ², even though they play an important role in the validity of the GSL [8–10]. In deed, we take into account the energy change in the box and the effects of thermal atmosphere and showed that these effects have an important role to prevent the violation of the GSL.

² Thus, it is doubtful if their composite system can be considered to be a thermal one (in the sense of thermally contacted system). On the other hand, since we adiabatically lowers the box toward the black hole, this condition is naturally justified.

Of course, our analysis is not perfect: for instance, we have neglected interactions between the constituents of the radiation and the thermal atmosphere. However, we could say that our analysis improves the previous analyses, even if we have not resolved all the difficulties.

Acknowledgments

One of us (T.S.) would like to thank Dr. T. Okamura for useful discussions, Professors A. Hosoya, H. Ishihara and T. Mishima for their continuing encouragement. The other (S.M.) would like to thank Professor H. Kodama for his continuing encouragement and Professor W. Israel for his warmest hospitality in University of Victoria and careful reading of the manuscript. This work was supported partially (S.M.) by the Grant-in-Aid for Scientific Research Fund (No. 9809228).

APPENDIX A: INTEGRATED GIBBS-DUHEM RELATION

Providing that the system is in equilibrium states, the first law of thermodynamics is

$$d\mathcal{E} = \tilde{T}d\mathcal{S} - \tilde{P}d\mathcal{V} + \mathcal{Q}d\phi + \sum_i \tilde{\mu}_i d\mathcal{N}_i, \quad (\text{A1})$$

where \mathcal{E} , \tilde{T} , \mathcal{S} , \tilde{P} , \mathcal{V} , \mathcal{Q} , ϕ , $\tilde{\mu}_i$ and \mathcal{N}_i are energy measured by a local static observer, local temperature, entropy, pressure, volume, electromagnetic charge, electromagnetic potential, chemical potential and particle number density, respectively.

Since

$$d(\mathcal{E} - \mathcal{Q}\phi) = \tilde{T}d\mathcal{S} - \tilde{P}d\mathcal{V} - \phi d\mathcal{Q} + \sum_i \tilde{\mu}_i d\mathcal{N}_i, \quad (\text{A2})$$

the quantity $\mathcal{E} - \mathcal{Q}\phi$ is a function of \mathcal{S} , \mathcal{V} , \mathcal{Q} and \mathcal{N}_i :

$$\mathcal{E} - \mathcal{Q}\phi = \mathcal{F}(\mathcal{S}, \mathcal{V}, \mathcal{Q}, \mathcal{N}_i). \quad (\text{A3})$$

Thus, since \mathcal{F} is homogeneous function of degree 1 in these extensive parameters, we get

$$\alpha(\mathcal{E} - \mathcal{Q}\phi) = \mathcal{F}(\alpha\mathcal{S}, \alpha\mathcal{V}, \alpha\mathcal{Q}, \alpha\mathcal{N}_i). \quad (\text{A4})$$

By differentiating this equation with respect to α and setting $\alpha = 1$, we obtain

$$\mathcal{E} - \mathcal{Q}\phi = \left(\frac{\partial \mathcal{F}}{\partial \mathcal{S}}\right)_{\mathcal{V}, \mathcal{Q}, \mathcal{N}_i} \mathcal{S} + \left(\frac{\partial \mathcal{F}}{\partial \mathcal{V}}\right)_{\mathcal{S}, \mathcal{Q}, \mathcal{N}_i} \mathcal{V} + \left(\frac{\partial \mathcal{F}}{\partial \mathcal{Q}}\right)_{\mathcal{S}, \mathcal{V}, \mathcal{N}_i} \mathcal{Q} + \sum_i \left(\frac{\partial \mathcal{F}}{\partial \mathcal{N}_i}\right)_{\mathcal{S}, \mathcal{V}, \mathcal{Q}} \mathcal{N}_i. \quad (\text{A5})$$

Therefore, we get the integrated Gibbs-Duhem relation:

$$\mathcal{E} = \tilde{T}\mathcal{S} - \tilde{P}\mathcal{V} + \sum_i \tilde{\mu}_i \mathcal{N}_i. \quad (\text{A6})$$

Eqs.(A1) and (A6) also can be rewritten as the relations between local quantities:

$$\begin{aligned} d\tilde{\rho} &= \tilde{T}d\tilde{s} + \tilde{q}d\phi + \sum_i \tilde{\mu}_i d\tilde{n}_i, \\ \tilde{\rho} &= \tilde{T}\tilde{s} - \tilde{P} + \sum_i \tilde{\mu}_i \tilde{n}_i. \end{aligned} \quad (\text{A7})$$

where $\tilde{\rho}$ ($= \mathcal{E}/\mathcal{V}$), \tilde{s} ($= \mathcal{S}/\mathcal{V}$), \tilde{q} ($= \mathcal{Q}/\mathcal{V}$) and \tilde{n}_i ($= \mathcal{N}_i/\mathcal{V}$) are energy density measured by a local static observer, entropy density, charge density and number density, respectively.

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